

A SIMPLY CONNECTED SURFACE OF GENERAL TYPE WITH $p_g = 0$ AND $K^2 = 2$

YONGNAM LEE AND JONGIL PARK

ABSTRACT. In this paper we construct a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 2$ using a rational blow-down surgery and a \mathbb{Q} -Gorenstein smoothing theory.

1. INTRODUCTION

One of the fundamental problems in the classification of complex surfaces is to find a new family of simply connected surfaces of general type with $p_g = 0$. Surfaces with $p_g = 0$ are interesting in view of Castelnuovo's criterion: An irrational surface with $q = 0$ must have $P_2 \geq 1$. This class of surfaces has been studied extensively by algebraic geometers and topologists for a long time. The details of history and examples of surfaces with $p_g = 0$ are given in [8]. In particular, simply connected surfaces of general type with $p_g = 0$ are little known and the classification is still open. Therefore it is a very important problem to find a new family of simply connected surfaces with $p_g = 0$. Although a large number of non-simply connected complex surfaces of general type with $p_g = 0$ have been known ([4], Chapter VII), until now the only previously known simply connected, minimal, complex surface of general type with $p_g = 0$ was Barlow surface [3]. Barlow surface has $K^2 = 1$. The natural question arises if there is a simply connected surface of general type with $p_g = 0$ and $K^2 \geq 2$.

Recently, the second author constructed a new simply connected symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 2$ using a rational blow-down surgery [34]. After this construction, it has been a very intriguing question whether such a symplectic 4-manifold admits a complex structure. On the other hand, the compactification theory of a moduli space of surfaces of general type was established during the last twenty years. It was originally suggested in [19] and it was established by Alexeev's proof for boundness [1] and by the Mori program for threefolds ([18] for details). It is natural to expect the existence of a surface with special quotient singularities in the boundary of a compact moduli space. Hence it would be possible to construct a new interesting surface of general type by using a singular surface with special quotient singularities. This was also more or less suggested by Kollár in a recent paper [17].

The aim of this paper is to give an affirmative answer for the question above. Precisely, we construct a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 2$ by modifying Park's symplectic 4-manifold in [34]. Our approach is very different from other classical constructions such as a finite group quotient and a double covering, due to Godeaux, Campedelli, Burniat and others. These classical

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constructions are well explained in [36]. Our main techniques are rational blow-down surgery and \mathbb{Q} -Gorenstein smoothing theory. These theories are briefly reviewed and developed in Section 2, 3 and 4. We first consider a special cubic pencil in \mathbb{P}^2 and blow up many times to get a projective surface \tilde{Z} which contains several disjoint chains of curves representing the resolution graphs of special quotient singularities. Then we contract these chains of curves from the surface \tilde{Z} to produce a projective surface X with five special quotient singularities. The details of the construction are given in Section 3. Using the methods developed in Section 2, we prove in Section 4 that the singular surface X has a \mathbb{Q} -Gorenstein smoothing. And we also prove in Section 5 that the general fiber X_t of the \mathbb{Q} -Gorenstein smoothing is a minimal surface of general type with $p_g = 0$ and $K^2 = 2$. Finally, applying the standard arguments about Milnor fibers ([23] §5, or [26] §1), we prove that the general fiber X_t is diffeomorphic to a simply connected symplectic 4-manifold $\tilde{Z}_{15,9,5,3,2}$ which is obtained by rationally blowing down along five disjoint configurations in \tilde{Z} . The main result of this paper is the following

Theorem 1.1. *There exists a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 2$.*

We provide more examples of simply connected surfaces of general type with $p_g = 0$ and $K^2 = 2$ in Section 6. By using a different configuration, we are also able to construct a minimal, complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}_2$ [21]. Furthermore, by a small modification of the main construction and using the same techniques, we also construct simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $K^2 = 1$. These constructions are explained in Appendix.

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2. \mathbb{Q} -GORENSTEIN SMOOTHING

In this section we develop a theory of \mathbb{Q} -Gorenstein smoothing for projective surfaces with special quotient singularities, which is a key technical ingredient in our result.

Definition. Let X be a normal projective surface with quotient singularities. Let $\mathcal{X} \rightarrow \Delta$ (or \mathcal{X}/Δ) be a flat family of projective surfaces over a small disk Δ . The one-parameter family of surfaces $\mathcal{X} \rightarrow \Delta$ is called a *\mathbb{Q} -Gorenstein smoothing* of X if it satisfies the following three conditions;

- (i) the general fiber X_t is a smooth projective surface,

- (ii) the central fiber X_0 is X ,
- (iii) the canonical divisor $K_{X/\Delta}$ is \mathbb{Q} -Cartier.

A \mathbb{Q} -Gorenstein smoothing for a germ of a quotient singularity $(X_0, 0)$ is defined similarly. A quotient singularity which admits a \mathbb{Q} -Gorenstein smoothing is called a *singularity of class T* .

Proposition 2.1 ([19, 25, 41]). *Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a \mathbb{Q} -Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna - 1)$ for some integers a, n, d with a and n relatively prime.*

Proposition 2.2 ([19, 25, 42]). (1) *The singularities $\overset{-4}{\circ}$ and $\overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \dots - \overset{-2}{\circ} - \overset{-3}{\circ}$ are of class T .*

(2) *If the singularity $\overset{-b_1}{\circ} - \dots - \overset{-b_r}{\circ}$ is of class T , then so are*

$$\overset{-2}{\circ} - \overset{-b_1}{\circ} - \dots - \overset{-b_{r-1}}{\circ} - \overset{-b_r-1}{\circ} \quad \text{and} \quad \overset{-b_1-1}{\circ} - \overset{-b_2}{\circ} - \dots - \overset{-b_r}{\circ} - \overset{-2}{\circ}.$$

(3) *Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2).*

Let X be a normal projective surface with singularities of class T . Due to the result of Kollár and Shepherd-Barron [19], there is a \mathbb{Q} -Gorenstein smoothing locally for each singularity of class T on X (see Proposition 2.5). The natural question arises whether this local \mathbb{Q} -Gorenstein smoothing can be extended over the global surface X or not. Roughly geometric interpretation is the following: Let $\cup V_\alpha$ be an open covering of X such that each V_α has at most one singularity of class T . By the existence of a local \mathbb{Q} -Gorenstein smoothing, there is a \mathbb{Q} -Gorenstein smoothing $\mathcal{V}_\alpha/\Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T_X^i = \text{Ext}_X^i(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if X is a smooth surface, then T_X^0 is the usual holomorphic tangent sheaf T_X and $T_X^1 = T_X^2 = 0$. By applying the standard result of deformations [22, 32] to a normal projective surface with quotient singularities, we get the following

Proposition 2.3 ([41], §4). *Let X be a normal projective surface with quotient singularities. Then*

- (1) *The first order deformation space of X is represented by the global Ext 1-group $\mathbb{T}_X^1 = \text{Ext}_X^1(\Omega_X, \mathcal{O}_X)$.*
- (2) *The obstruction lies in the global Ext 2-group $\mathbb{T}_X^2 = \text{Ext}_X^2(\Omega_X, \mathcal{O}_X)$.*

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves ([32], §3) to deformation theory of surfaces with quotient singularities so that $E_2^{p,q} = H^p(T_X^q) \Rightarrow \mathbb{T}_X^{p+q}$, and by $H^j(T_X^i) = 0$ for $i, j \geq 1$, we also get

Proposition 2.4 ([25, 41]). *Let X be a normal projective surface with quotient singularities. Then*

- (1) *We have the exact sequence*

$$0 \rightarrow H^1(T_X^0) \rightarrow \mathbb{T}_X^1 \rightarrow \ker[H^0(T_X^1) \rightarrow H^2(T_X^0)] \rightarrow 0$$

where $H^1(T_X^0)$ represents the first order deformations of X for which the singularities remain locally a product.

(2) If $H^2(T_X^0) = 0$, every local deformation of the singularities may be globalized.

Remark. The vanishing $H^2(T_X^0) = 0$ can be obtained via the vanishing of $H^2(T_V(-\log E))$, where V is the minimal resolution of X and E is the reduced exceptional divisors.

Let V be a nonsingular surface and let D be a simple normal crossing divisor in V . The short exact sequence

$$0 \rightarrow \mathcal{O}_D(-D) \rightarrow \Omega_V|_D \rightarrow \Omega_D \rightarrow 0$$

induces

$$0 \rightarrow T_D^0 \rightarrow T_V|_D \rightarrow \mathcal{O}_D(D) \rightarrow \text{Ext}^1(\Omega_D, \mathcal{O}_D) \rightarrow 0.$$

Hence there is a short exact sequence

$$0 \rightarrow T_D^0 \rightarrow T_V|_D \rightarrow \oplus_i N_{D_i|V} \rightarrow 0$$

where D_i is a component of D . Then the sheaf $T_V(-\log D)$ is defined by the following natural exact sequence

$$0 \rightarrow T_V(-\log D) \rightarrow T_V \rightarrow \oplus_i N_{D_i|V} \rightarrow 0.$$

M. Manetti provided the following lemma for us and he also suggested a simple and direct proof of Theorem 2.1 below [24]. Our original approach is through a theory of global smoothings of varieties with normal crossings developed by R. Friedman [13].

Lemma 2.1 ([24]). *Let $\pi : (V, E) \rightarrow (X_0, 0)$ be the minimal resolution of a germ of two dimensional quotient singularity and let $E = \cup E_i$ be the reduced exceptional divisor. Then $R^1\pi_*T_V(-\log E) = R^2\pi_*T_V(-\log E) = 0$.*

Proof. We may assume that X_0 is affine. For every effective divisor Z supported in E , its tangent sheaf Θ_Z fits into the commutative diagram ([5], p.70):

$$\begin{array}{ccccccc} 0 & \rightarrow & T_V(-\log E) & \rightarrow & T_V & \rightarrow & \oplus_i N_{E_i|V} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \Theta_Z & \rightarrow & T_V \otimes \mathcal{O}_Z & \rightarrow & \oplus_i N_{E_i|V} \rightarrow 0. \end{array}$$

Snake lemma gives the exact sequence

$$0 \rightarrow T_V(-Z) \rightarrow T_V(-\log E) \rightarrow \Theta_Z \rightarrow 0$$

and then, for Z sufficiently big, we get $H^1(T_V(-\log E)) = H^1(\Theta_Z)$: If F is a locally free sheaf on V , then $H^2(V, F) = 0$ and, since the singularity is rational, we also get $H^1(V, F(-Z)) = 0$. The proof of two facts follows easily from ([4], p.93–95), and it is sufficient that, for every component E_i , the restriction of $F(-Z)$ is a direct sum $\oplus_i \mathcal{O}_{E_i}(a_i)$ with all $a_i \geq 0$. According to the tautness of quotient singularities [20], we have $H^1(\Theta_Z) = 0$ for every Z sufficiently large. \square

Theorem 2.1. *Let X be a normal projective surface with singularities of class T . Let $\pi : V \rightarrow X$ be the minimal resolution and let E be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log E)) = 0$. Then there is a \mathbb{Q} -Gorenstein smoothing of X .*

Proof. ([24]) By Lemma 2.1 above, $R^1\pi_*T_V(-\log E) = R^2\pi_*T_V(-\log E) = 0$ and then, Leray spectral sequence implies

$$H^2(\pi_*T_V(-\log E)) = H^2(T_V(-\log E)) = 0.$$

Since $\pi_*T_V = T_X^0$ [5], we have

$$0 \rightarrow \pi_*T_V(-\log E) \rightarrow T_X^0 \rightarrow \Delta \rightarrow 0,$$

where Δ is supported in the singular locus of X . Therefore we get

$$H^2(T_X^0) = H^2(\pi_*T_V(-\log E)) = 0.$$

Hence every local deformation of the singularities may be globalized by Proposition 2.4. Furthermore, there is a \mathbb{Q} -Gorenstein smoothing of X by Proposition 2.5 below. \square

Let X be a normal projective surface with singularities of class T . Our concern is to understand \mathbb{Q} -Gorenstein smoothings in T_X^1 , not the whole first order deformations. These special deformations can be constructed via local index one cover. Let $U \subset X$ be an analytic neighborhood with an index one cover U' . For the case of the field \mathbb{C} , this index one cover is unique up to isomorphism. The first order deformations which associate \mathbb{Q} -Gorenstein smoothings can be realized as the invariant part of $T_{U'}^1$. By the help of the birational geometry in threefolds and their applications to deformations of surface singularities, the following proposition is obtained.

Proposition 2.5 ([19]). *Let $(X_0, 0)$ be a germ of singularity of class T . Then $(X_0, 0)$ admits a \mathbb{Q} -Gorenstein smoothing.*

Note that Theorem 2.1 above can be easily generalized to any log resolution of X by keeping the vanishing of cohomologies under blowing up at the points. It is obtained by the following well-known result. Proposition 2.6 is also used in Section 4.

Proposition 2.6 ([12], §1). *Let V be a nonsingular surface and let D be a simple normal crossing divisor in V . Let $f : V' \rightarrow V$ be a blowing up of V at a point p of D . Set $D' = f^{-1}(D)_{\text{red}}$. Then $h^2(T_{V'}(-\log D')) = h^2(T_V(-\log D))$.*

Example. We consider a pencil of cubics in \mathbb{P}^2 and blow up at the base points. Denote this surface by Y . Then Y has an elliptic fibration over \mathbb{P}^1 . Assume that there is a nodal fiber. Let $\tau : Z \rightarrow Y$ be a blowing-up at the singular point on this nodal fiber. Let F be the proper transform of the nodal fiber and let E be the exceptional curve in Z . We construct a projective surface X with one singularity of class T by contracting -4 -curve F . Since $H^2(Z, T_Z(-F)) = 0$, Theorem 2.1 implies that X has a \mathbb{Q} -Gorenstein smoothing. Using the fact that $-K_X$ is an effective divisor together with the result of Manetti ([25], Theorem 21), one can also prove the existence of a \mathbb{Q} -Gorenstein smoothing of X . Using the result again in [25], it is not hard to show that a general fiber X_t of a \mathbb{Q} -Gorenstein smoothing is a smooth rational elliptic surface with $K_{X_t}^2 = 0$.

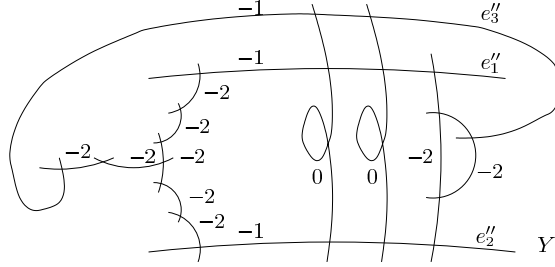
Remark. Gompf [14] constructed a symplectic 4-manifold by taking a fiber sum of two symplectic 4-manifolds. To briefly recall Gompf's example, we start with a simply connected relatively minimal elliptic surface with a section and with $c_2 = 48$. There is only one up to diffeomorphism such an elliptic surface, which is called $E(4)$. It is also known that $E(4)$ admits nine rational (-4) -curves as disjoint sections. Rationally blowing down n (-4) -curves of $E(4)$ is the same as the normal connected sum of $E(4)$

with n copies of \mathbb{P}^2 by identifying a conic in each \mathbb{P}^2 with one (-4) -curve in $E(4)$. This 4-manifold is denoted by $W_{4,n}$. The manifold $W_{4,1}$ does not admit any complex structure because it violates the Noether inequality $p_g \leq \frac{1}{2}K^2 + 2$ (cf. [4]). In fact, we have $H^2(E(4), T_{E(4)}) \neq 0$. Therefore it does not satisfy the vanishing condition in Theorem 2.1: Let $h : E(4) \rightarrow \mathbb{P}^1$ be an elliptic fibration. Assume C is a general fiber of the map f . We have an injective map $0 \rightarrow h^*\Omega_{\mathbb{P}^1} \rightarrow \Omega_{E(4)}$ and the map induces an injection $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(2)) \hookrightarrow H^0(E(4), \Omega_{E(4)}(2C))$ by tensoring $2C$ on $0 \rightarrow h^*\Omega_{\mathbb{P}^1} \rightarrow \Omega_{E(4)}$. Since $K_{E(4)} = 2C$, the cohomology $H^0(E(4), \Omega_{E(4)}(K_{E(4)}))$ is not zero. Hence the Serre duality implies that $H^2(E(4), T_{E(4)})$ is not zero. In fact, it is not hard to show that $h^2(E(4), T_{E(4)}) = 1$.

3. THE MAIN CONSTRUCTION

We begin with the rational elliptic surface $E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}^2}$. There are several ways to construct an elliptic fibration on $E(1)$. In this paper we use a special elliptic fibration $g : E(1) \rightarrow \mathbb{P}^1$ which is constructed as follows: Let A be a line and B be a smooth conic in \mathbb{P}^2 which are represented homologically by h and $2h$ respectively, where h denotes a generator of $H_2(\mathbb{P}^2; \mathbb{Z})$. Choose another line L in \mathbb{P}^2 which meets B at two distinct points p, q , and which also meets A at a different point r . We may assume that the conic B and the line A meet at two different points which are not p, q, r . We now consider a cubic pencil in \mathbb{P}^2 induced by $A + B$ and $3L$, i.e. $\lambda(A + B) + \mu(3L)$, for $[\lambda : \mu] \in \mathbb{P}^1$. After we blow up first at three points p, q, r , blow up again three times at the intersection points of the proper transforms of B, A with the three exceptional curves e_1, e_2, e_3 . Finally, blowing up again three times at the intersection points of the proper transforms of B (representing homologically $2h - e_1 - e_2 - e'_1 - e'_2$) and A (representing homologically $h - e_3 - e'_3$) with the three new exceptional curves e'_1, e'_2, e'_3 , we get an elliptic fibration $E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}^2}$ over \mathbb{P}^1 . We denote the three new exceptional curves by e''_1, e''_2, e''_3 , and let us denote this elliptic fibration by $g : Y = E(1) \rightarrow \mathbb{P}^1$. We note that there is an \tilde{E}_6 -singular fiber (IV^* in Kodaira's table of singular elliptic fibers in [4], p.201) on the fibration $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of $L, e_1, e'_1, e_2, e'_2, e_3, e'_3$. We also note that there is one I_2 -singular fiber (two rational -2 -curves meeting two points) on $g : Y \rightarrow \mathbb{P}^1$ which consists of the proper transforms of the line A and the conic B . Furthermore, by the proper choice of curves A, B and L guarantees two more nodal singular fibers on $g : Y \rightarrow \mathbb{P}^1$. For example, the pencil $\lambda(xy - z^2)z + \mu(x - y)^3$ works. This pencil has singular fibers at $[\lambda : \mu] = [1 : 0], [0 : 1], [36 : \sqrt{3}i]$ and $[-36 : \sqrt{3}i]$. Hence the fibration $g : Y \rightarrow \mathbb{P}^1$ has one \tilde{E}_6 -singular fiber, one reducible I_2 -singular fiber, and two nodal singular fibers (see Figure 1 below). Notice that there are three sections e''_1, e''_2, e''_3 in Y , so that two sections e''_1, e''_2 meet the proper transform of the conic B and the third section e''_3 meets the proper transform of the line A .

Remark. The existence of an elliptic fibration with fibres I_1, I_1, I_2 and IV^* is known ([15, 35]). And, by the result in [30], this elliptic fibration has three sections which satisfy the configuration in Figure 1: Let C_1, C_2, C_3 be three ends of IV^* and D_1, D_2 the two component of I_2 . We number as the zero section O passes C_1 and D_1 . By the height formula, the generator P of the Mordell Weil group passes C_2 (after changing C_2 and C_3 if necessary) and D_2 . Then the section corresponding to $2P$ passes C_3 and D_1 . Thus $O, 2P$ and P satisfy the configuration [29].

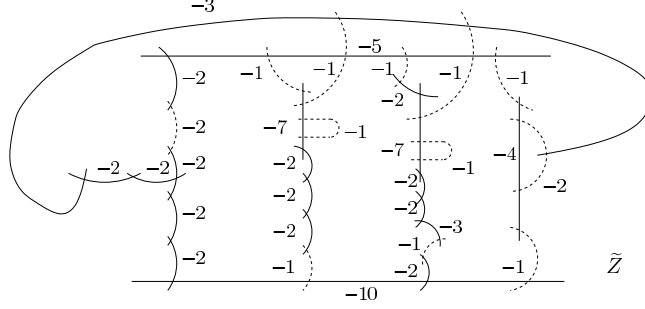
FIGURE 1. A rational surface Y

Main Construction. Let $Z := Y \# 2\overline{\mathbb{P}}^2$ be the surface obtained by blowing up at two singular points of two nodal fibers on Y , and denote this map by τ . Then there are two fibers such that each consists of two \mathbb{P}^1 s, say E_i and F_i , satisfying $E_i^2 = -1$, $F_i^2 = -4$ and $E_i \cdot F_i = 2$ for $i = 1, 2$. Note that each E_i is an exceptional curve and F_i is the proper transform of a nodal fiber. Then the surface Z has four special fibers; one \tilde{E}_6 -singular fiber, one I_2 -singular fiber consisting of \tilde{A} and \tilde{B} which are the proper transforms of A and of B , and two more singular fibers which are the union of E_i and F_i for $i = 1, 2$. We denote the three sections e''_1, e''_2, e''_3 by S_1, S_2, S_3 respectively. First, we blow up six times at the intersection points between two sections S_1, S_2 and F_1, F_2, \tilde{B} . It makes the self-intersection number of the proper transforms of S_1, S_2 and \tilde{B} to be -4 . We also blow up twice at the intersection points between the third section S_3 and F_1, F_2 , so that the self-intersection number of the proper transform of S_3 is -3 .

Next, we blow up three times successively at the intersection point between the proper transform of S_2 and the exceptional curve in the total transform of F_1 . It makes a chain of \mathbb{P}^1 , $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_1 . We also blow up three times successively at the intersection point between the proper transform of S_2 and the exceptional curve in the total transform of F_2 , so that a chain of \mathbb{P}^1 , $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lies in the total transform of F_2 . We note that the self-intersection numbers of the proper transforms of F_1 and F_2 are -7 . Then we blow up at the intersection point between the proper transform of S_1 and the exceptional -1 -curve intersecting the proper transform of F_2 , so that it produces a chain of \mathbb{P}^1 , $\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_2 . Then we blow up again at the intersection point between the exceptional -1 -curve and the rational -2 -curve which is the right end of the above chain of \mathbb{P}^1 , so that it produces a chain of \mathbb{P}^1 , $\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}$. We note that the self-intersection numbers of the proper transforms of S_1 and S_2 go to -5 and -10 respectively.

Next, we have a rational surface $\tilde{Z} := Y \# 18\overline{\mathbb{P}}^2$ which contains five disjoint linear chains of \mathbb{P}^1 : $\overset{-2}{\circ} - \overset{-10}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ (which contains the proper transforms of two sections S_2, S_3 and the five rational -2 -curves in \tilde{E}_6 -singular fiber), $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, $\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$, $\overset{-4}{\circ}$ and $\overset{-5}{\circ} - \overset{-2}{\circ}$ (which contains the proper transforms of the section S_1 and the one rational -2 -curve in \tilde{E}_6 -singular fiber) (Figure 2).

Finally, we contract these five disjoint chains of \mathbb{P}^1 from \tilde{Z} . Since it satisfies the Artin's criterion, it produces a projective surface with five singularities of class T ([2], §2). We denote this surface by X . In Section 4 we will prove that X has a \mathbb{Q} -Gorenstein

FIGURE 2. A rational surface \tilde{Z}

smoothing. And we will also show in Section 5 that a general fiber of the \mathbb{Q} -Gorenstein smoothing is a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 2$.

In the remaining of this section, we investigate a rational blow-down manifold of the surface \tilde{Z} . First we describe topological aspects of a rational blow-down surgery ([11, 33] for details): For any relatively prime integers p and q with $p > q > 0$, we define a configuration $C_{p,q}$ as a smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere instructed by the following linear diagram

$$\begin{array}{ccccccc} -b_k & -b_{k-1} & & -b_2 & -b_1 \\ \circ & \circ & \cdots & \circ & \circ \\ u_k & u_{k-1} & & u_2 & u_1 \end{array}$$

where $\frac{p^2}{pq-1} = [b_k, b_{k-1}, \dots, b_1]$ is the unique continued fraction with all $b_i \geq 2$, and each vertex u_i represents a disk bundle over the 2-sphere whose Euler number is $-b_i$. Orient the 2-spheres in $C_{p,q}$ so that $u_i \cdot u_{i+1} = +1$. Then the configuration $C_{p,q}$ is a negative definite simply connected smooth 4-manifold whose boundary is the lens space $L(p^2, 1 - pq)$.

Definition. Suppose M is a smooth 4-manifold containing a configuration $C_{p,q}$. Then we construct a new smooth 4-manifold M_p , called a (*generalized*) *rational blow-down* of M , by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. Note that this process is well-defined, that is, a new smooth 4-manifold M_p is uniquely determined (up to diffeomorphism) from M because each diffeomorphism of $\partial B_{p,q}$ extends over the rational ball $B_{p,q}$. We call this a *rational blow-down* surgery. Furthermore, M. Symington proved that a rational blow-down manifold M_p admits a symplectic structure in some cases. For example, if M is a symplectic 4-manifold containing a configuration $C_{p,q}$ such that all 2-spheres u_i in $C_{p,q}$ are symplectically embedded and intersect positively, then the rational blow-down manifold M_p also admits a symplectic structure [39, 40].

Now we perform a rational blow-down surgery on the surface \tilde{Z} constructed in the main construction. Note that the surface \tilde{Z} contains five disjoint configurations - $C_{15,7}$, $C_{9,4}$, $C_{5,1}$, $C_{3,1}$ and $C_{2,1}$. Let us decompose the surface \tilde{Z} into

$$\tilde{Z} = \tilde{Z}_0 \cup \{C_{15,7} \cup C_{9,4} \cup C_{5,1} \cup C_{3,1} \cup C_{2,1}\}.$$

Then the 4-manifold, say $\tilde{Z}_{15,9,5,3,2}$, obtained by rationally blowing down along the five configurations can be decomposed into

$$\tilde{Z}_{15,9,5,3,2} = \tilde{Z}_0 \cup \{B_{15,7} \cup B_{9,4} \cup B_{5,1} \cup B_{3,1} \cup B_{2,1}\},$$

where $B_{15,7}$, $B_{9,4}$, $B_{5,1}$, $B_{3,1}$ and $B_{2,1}$ are the corresponding rational balls. We claim that

Theorem 3.1. *The rational blow-down $\tilde{Z}_{15,9,5,3,2}$ of the surface \tilde{Z} is a simply connected closed symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 2$.*

Proof. Since all the curves lying in the configurations $C_{15,7}$, $C_{9,4}$, $C_{5,1}$, $C_{3,1}$ and $C_{2,1}$ are symplectically (in fact, holomorphically) embedded 2-spheres, Symington's result [39, 40] guarantees the existence of a symplectic structure on the rational blow-down 4-manifold $\tilde{Z}_{15,9,5,3,2}$. Furthermore, it is easy to check that $b_2^+(\tilde{Z}_{15,9,5,3,2}) = b_2^+(\tilde{Z}) = 1$ and $K^2(\tilde{Z}_{15,9,5,3,2}) = K^2(\tilde{Z}) + 20 = 2$.

It remains to prove the simple connectivity of $\tilde{Z}_{15,9,5,3,2}$: Since $\pi_1(\partial B_{p,q}) \rightarrow \pi_1(B_{p,q})$ is surjective ([23], §5), by Van-Kampen theorem, it suffices to show that $\pi_1(\tilde{Z}_0) = 1$. First, note that \tilde{Z} and all four configurations $C_{15,7}$, $C_{9,4}$, $C_{5,1}$, $C_{3,1}$ and $C_{2,1}$ are all simply connected. Hence, applying Van-Kampen theorem on \tilde{Z} inductively, we get

$$(1) \quad 1 = \pi_1(\tilde{Z}_0) / \langle N_{i_*(\alpha)}, N_{j_{1*}(\beta_1)}, N_{j_{2*}(\beta_2)}, N_{j_{3*}(\beta_3)}, N_{k_*(\gamma)} \rangle.$$

Here i_* , j_{1*} , j_{2*} , j_{3*} and k_* are induced homomorphisms by inclusions $i : \partial C_{3,1} \rightarrow \tilde{Z}_0$, $j_1 : \partial C_{5,1} \rightarrow \tilde{Z}_0$, $j_2 : \partial C_{9,4} \rightarrow \tilde{Z}_0$, $j_3 : \partial C_{2,1} \rightarrow \tilde{Z}_0$ and $k : \partial C_{15,7} \rightarrow \tilde{Z}_0$ respectively. We may also choose the generators, say α , β_1 , β_2 , β_3 and γ , of $\pi_1(\partial C_{3,1}) \cong \mathbb{Z}_9$, $\pi_1(\partial C_{5,1}) \cong \mathbb{Z}_{25}$, $\pi_1(\partial C_{9,4}) \cong \mathbb{Z}_{81}$, $\pi_1(\partial C_{2,1}) \cong \mathbb{Z}_4$ and $\pi_1(\partial C_{15,7}) \cong \mathbb{Z}_{225}$, so that α , β_1 , β_2 , β_3 and γ are represented by circles $\partial C_{3,1} \cap E'_1$ (equivalently $\partial C_{3,1} \cap E'_2$ or $\partial C_{3,1} \cap E'_3$), $\partial C_{5,1} \cap E'_1$, $\partial C_{9,4} \cap E'_2$ (equivalently $\partial C_{9,4} \cap E'_2$), $\partial C_{2,1} \cap E'_3$ and $\partial C_{15,7} \cap E'_2$, respectively, where E'_1 , E'_2 , E'_3 and E'_2 are exceptional curves connecting the last 2-spheres in the configurations $C_{3,1}$ and $C_{5,1}$, $C_{3,1}$ and $C_{9,4}$, $C_{3,1}$ and $C_{2,1}$, $C_{15,7}$ and $C_{9,4}$, respectively. Note that the circle cut out by a 2-sphere which intersects transversely one of the two end 2-spheres in the configurations $C_{p,q}$ is a generator of π_1 of the lens space, and other circles cut out by a 2-sphere which intersects transversely one of the middle 2-spheres in the configurations $C_{p,q}$, is a power of the generator [27]. Finally $N_{i_*(\alpha)}$, $N_{j_{i*}(\beta_i)}$ and $N_{k_*(\gamma)}$ denote the least normal subgroups of $\pi_1(\tilde{Z}_0)$ containing $i_*(\alpha)$, $j_{i*}(\beta_i)$ and $k_*(\gamma)$ respectively. Note that there is a relation between $i_*(\alpha)$ and $j_{i*}(\beta_i)$ when we restrict them to \tilde{Z}_0 . That is, they satisfy either $i_*(\alpha) = \tau^{-1} \cdot j_{i*}(\beta_i) \cdot \tau$ or $i_*(\alpha) = \tau^{-1} \cdot j_{i*}(\beta_i)^{-1} \cdot \tau$ (depending on orientations) for some path τ , because one is homotopic to the other in $E'_i \setminus \{\text{two open disks}\} \subset \tilde{Z}_0$. Hence, by combining two facts above, we get $i_*(\alpha)^9 = (\tau^{-1} \cdot j_{3*}(\beta_3)^{\pm 1} \cdot \tau)^9 = \tau^{-1} \cdot j_{3*}(\beta_3)^{\pm 9} \cdot \tau = 1 = j_{3*}(\beta_3)^4$. Since the two numbers 9 and 4 are relatively prime, the element $j_{3*}(\beta_3)$ should be trivial. So the relation $i_*(\alpha) = \tau^{-1} \cdot j_{3*}(\beta_3)^{\pm 1} \cdot \tau$ implies the triviality of $i_*(\alpha)$. And the triviality of $i_*(\alpha)$ also implies that $j_{1*}(\beta_1)$ and $j_{2*}(\beta_2)$ are trivial. Furthermore, since $j_{2*}(\beta_2)$ and $k_*(\gamma)$ are also conjugate each other, $k_*(\gamma)$ is also trivial. Hence, all normal subgroups $N_{i_*(\alpha)}$, $N_{j_{i*}(\beta_i)}$ and $N_{k_*(\gamma)}$ are trivial, so that relation (1) implies $\pi_1(\tilde{Z}_0) = 1$. \square

Remark. In fact, one can prove that the rational blow-down 4-manifold $\tilde{Z}_{15,9,5,3,2}$ constructed in Theorem 3.1 above is not diffeomorphic to a rational surface $\mathbb{P}^2 \# 7\overline{\mathbb{P}}^2$ by

using a technique in [34]. Furthermore, one can also prove that the symplectic 4-manifold $\tilde{Z}_{15,9,5,3,2}$ is minimal by using a similar technique in [31].

4. EXISTENCE OF SMOOTHING

In this section we prove the existence of a \mathbb{Q} -Gorenstein smoothing for the main example constructed in Section 3.

Lemma 4.1. *Let Y be a rational elliptic surface. Let C be a general fiber of the elliptic fibration $g : Y \rightarrow \mathbb{P}^1$. Then the global sections $H^0(Y, \Omega_Y(kC))$ are coming from the global sections $H^0(Y, g^*\Omega_{\mathbb{P}^1}(k))$. In particular, $h^0(Y, \Omega_Y(kC)) = k - 1$ for $k \geq 1$.*

Proof. We have an injective map $0 \rightarrow g^*\Omega_{\mathbb{P}^1} \rightarrow \Omega_Y$, and the map induces an injection $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(k)) \hookrightarrow H^0(Y, \Omega_Y(kC))$ by tensoring kC on $0 \rightarrow g^*\Omega_{\mathbb{P}^1} \rightarrow \Omega_Y$.

The dimension of the cohomology $H^0(Y, \Omega_Y(kC))$ is $k - 1$: Consider the commutative diagram of standard exact sequences

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{O}_C((k-1)C) & & & \\
 & & & \downarrow & & & \\
 0 \rightarrow \Omega_Y((k-1)C) \rightarrow \Omega_Y(kC) \rightarrow & \Omega_Y(kC)|_C & \rightarrow 0 & & & & \\
 & \downarrow & & & & & \\
 & \Omega_C(kC) & & & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & & .
 \end{array}$$

The map from the long exact sequence of cohomologies from the vertical sequence

$$H^0(C, \Omega_C(kC)) \simeq H^0(C, \Omega_C) \rightarrow H^1(C, \mathcal{O}_C((k-1)C)) \simeq H^1(C, N_{C|Y}^\vee)$$

is the dual of the Kodaira-Spencer map $H^0(C, N_{C|Y}) \rightarrow H^1(C, T_C)$ because $\Omega_C = \mathcal{O}_C$. This map is not zero since the fibration is non-trivial. Therefore we have that $H^0(Y, \Omega_Y(kC)|_C) = H^0(C, \mathcal{O}_C)$ and $h^0(Y, \Omega_Y(kC)) \leq h^0(Y, \Omega_Y((k-1)C)) + 1$. Furthermore, the map $H^0(C, \mathcal{O}_C) \rightarrow H^1(Y, \Omega_Y)$ is injective because it factors through the exact sequence

$$0 \rightarrow \Omega_Y \rightarrow \Omega_Y(\log C) \rightarrow \mathcal{O}_C \rightarrow 0,$$

and the map $H^0(C, \mathcal{O}_C) \rightarrow H^1(Y, \Omega_Y)$ is the first Chern class map. Therefore we obtain the vanishing $H^0(Y, \Omega_Y(C)) = 0$ and $h^0(Y, \Omega_Y(kC)) \leq k - 1$. Since $H^0(Y, \Omega_Y(kC))$ contains $H^0(Y, g^*(\Omega_{\mathbb{P}^1}(k)))$, the dimension of $H^0(Y, \Omega_Y(kC))$ is $k - 1$. It implies that all global sections of $\Omega_Y(kC)$ are coming from the global sections of $g^*(\Omega_{\mathbb{P}^1}(k)) = g^*(\mathcal{O}_{\mathbb{P}^1}(k-2))$. \square

Example. Let Y be a smooth rational elliptic surface constructed by blowing up the base points of a pencil of cubics. Assume that there are two nodal fibers. Let $\tau : Z \rightarrow Y$ be a blowing-up at the singular points p, q on two nodal fibers. Let F_1, F_2 be the proper transforms of two nodal fibers and let E_1, E_2 be the two exceptional divisors in Z . Then we contract two -4 -curves F_1, F_2 . It produces a projective surface X with two singularities of class T whose resolution graphs are both $\begin{smallmatrix} -4 \\ \circ \end{smallmatrix}$. Let $f : Z \rightarrow X$ be the contraction morphism. Then X has a \mathbb{Q} -Gorenstein smoothing: By Theorem 2.1, it is enough to

prove $H^2(Z, T_Z(-F_1 - F_2)) = 0$. It is equal to prove $H^0(Z, \Omega_Z(K_Z + F_1 + F_2)) = 0$ by the Serre duality. Since $-\tau^*K_Y = F_i + 2E_i$ for $i = 1$ and 2 , we have

$$2K_Z = \tau^*(2K_Y) + 2E_1 + 2E_2 = -F_1 - F_2.$$

Therefore $H^0(Z, \Omega_Z(K_Z + F_1 + F_2)) = H^0(Z, \Omega_Z(-K_Z)) = H^0(Z, \Omega_Z(\tau^*C - E_1 - E_2))$, where C is a general fiber of an elliptic fibration from Y to \mathbb{P}^1 . Since $H^0(Z, \Omega_Z(\tau^*C - E_1 - E_2)) \subseteq H^0(Z, \Omega_Z(\tau^*C)) = H^0(Y, \Omega_Y(C))$, it is zero by Lemma 4.1. It is not hard to see that a general fiber X_t of a \mathbb{Q} -Gorenstein smoothing is a minimal Enriques surface.

If we contract more than two proper transforms of nodal fibers in a rational elliptic surface, then it does not satisfy the vanishing condition in Theorem 2.1.

Proposition 4.1. *Let Y be a smooth rational elliptic surface. Assume that the elliptic fibration $g : Y \rightarrow \mathbb{P}^1$ is relatively minimal without multiple fibers and there are j nodal fibers with $j \geq 3$. Let $\tau : Z \rightarrow Y$ be a blowing-up at the singular points p_1, \dots, p_j on nodal fibers. Let F_1, \dots, F_j be the proper transforms of nodal fibers and let E_1, \dots, E_j be the exceptional divisors in Z . Then $H^2(Z, T_Z(-F_1 - \dots - F_j)) \neq 0$.*

Proof. Let $h : Z \rightarrow \mathbb{P}^1$ be an elliptic fibration, and let s_1, \dots, s_j be points on \mathbb{P}^1 corresponding to nodal fibers F_1, \dots, F_j . Then there is an injective map ([10], p.80–81)

$$0 \rightarrow h^*\Omega_{\mathbb{P}^1}(s_1 + \dots + s_j) \rightarrow \Omega_Z(\log(F_1 + \dots + F_j + E_1 + \dots + E_j)).$$

Let C' be a general fiber of the map h , and s be a point on \mathbb{P}^1 corresponding to a fiber C' . Since $K_Z = -C' + E_1 + \dots + E_j$, $h^0(Z, \Omega_Z(\log(F_1 + \dots + F_j + E_1 + \dots + E_j))(K_Z)) \geq h^0(Z, h^*\Omega_{\mathbb{P}^1}(s_1 + \dots + s_j)(-s)) = j - 2$. Therefore, if $j \geq 3$, the Serre duality implies that $H^2(Z, T_Z(-\log(F_1 + \dots + F_j + E_1 + \dots + E_j))) \neq 0$. Since $E_i^2 = -1$ for $i = 1, \dots, j$, it is also equal to $H^2(Z, T_Z(-\log(F_1 + \dots + F_j))) \neq 0$. Hence it implies $H^2(T_Z(-F_1 - \dots - F_j)) \neq 0$. \square

By Proposition 4.1 above, we need to choose a smooth rational elliptic surface with special fibers. This is one of the reasons why we consider a rational elliptic surface with four special singular fibers in Section 3.

Lemma 4.2. *Let $Z = Y \# 2\bar{\mathbb{P}}^2$ be the rational elliptic surface in the main construction. Let F be the proper transform of the conic B in Z , and E be the proper transform of the line A in Z . Let D be the reduced subscheme of the \tilde{E}_6 -singular fiber. Assume that D is not whole \tilde{E}_6 -singular fiber as a reduced scheme. Then $H^2(Z, T_Z(-F_1 - F_2 - F - D)) = 0$ and $H^2(Z, T_Z(-F_1 - F_2 - E - D)) = 0$.*

Proof. By the Serre duality, it is equal to prove $H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) = 0$. Let C be a general fiber in the elliptic fibration $g : Y \rightarrow \mathbb{P}^1$. Since $K_Z = \tau^*(-C) + E_1 + E_2$ and $\tau^*(C) = F_1 + 2E_1$, $H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) \subseteq H^0(Z, \Omega_Z(\tau^*(C) + F + D))$. Since F and D are not changed by the map τ , we have the same curves in Y . Then $H^0(Z, \Omega_Z(\tau^*(C) + F + D)) = H^0(Y, \Omega_Y(C + F + D))$ by the projection formula. We note that $\tau_*\Omega_Z = \Omega_Y$. Then the cohomology $H^0(Y, \Omega_Y(C + F + D))$ vanishes: We note that $H^0(Y, \Omega_Y(C + F + D)) = H^0(Y, \Omega_Y(3C - E - D'))$ with $D' + D = \tilde{E}_6$ -singular fiber. By Lemma 4.1, all global sections of $\Omega_Y(3C)$ are coming from the global sections of $g^*(\Omega_{\mathbb{P}^1}(3)) = g^*(\mathcal{O}_{\mathbb{P}^1}(1))$. But, if this global section vanishes on E and D' which lie on two different fibers, then it should be zero. Note that the dualizing sheaf of each fiber of the elliptic fibration is the structure sheaf of the fiber by using the adjunction formula.

Therefore we have the vanishing $H^2(Z, T_Z(-F_1 - F_2 - F - D)) = 0$. The vanishing $H^2(Z, T_Z(-F_1 - F_2 - E - D)) = 0$ is obtained by the same proof. \square

Theorem 4.1. *The projective surface X with five singularities of class T in the main construction has a \mathbb{Q} -Gorenstein smoothing.*

Proof. Let D be the reduced scheme of the \tilde{E}_6 -singular fiber minus the rational -2 -curve J in the main construction in Section 3. Note that the curve J is not contracted from \tilde{Z} to X . By Lemma 4.2, we have $H^2(Z, T_Z(-F_1 - F_2 - F - D)) = H^2(Z, T_Z(-\log(F_1 + F_2 + F + D))) = 0$. Let $D_Z = F_1 + F_2 + F + D + S_1 + S_2 + S_3$. Since the self-intersection number of the section is -1 , we still have the vanishing $H^2(Z, T_Z(-\log D_Z)) = 0$. We blow up eight times at the intersection points between three sections (S_1, S_2 and S_3) and two nodal fibers, and at the intersection points between two sections (S_1 and S_2) and one I_2 -fiber. Denote this surface by Z' . Choose the exceptional curve in the total transform of F_2 which intersects the proper transform of S_1 , and choose two exceptional curves in the total transforms of F_1 and F_2 which intersect the proper transform of S_2 . Let $D_{Z'}$ be the reduced scheme of $F_1 + F_2 + F + D + S_1 + S_2 + S_3$ + these three exceptional divisors. Then, by Lemma 4.2, Proposition 2.6, and the self-intersection number, -1 , of each exceptional divisor, we have $H^2(Z', T_{Z'}(-\log D_{Z'})) = 0$. Finally, by using the same argument finite times through blowing up, we have the vanishing $H^2(\tilde{Z}, T_{\tilde{Z}}(-\log D_{\tilde{Z}})) = 0$, where $D_{\tilde{Z}}$ are the five disjoint linear chains of \mathbb{P}^1 which are the exceptional divisors from the contraction from \tilde{Z} to X . Hence there is a \mathbb{Q} -Gorenstein smoothing for X by Theorem 2.1. \square

5. PROPERTIES OF X_t

We showed in Section 4 that the projective surface X has a \mathbb{Q} -Gorenstein smoothing. We denote a general fiber of the \mathbb{Q} -Gorenstein smoothing by X_t . In this section, we prove that X_t is a simply connected, minimal, surface of general type with $p_g = 0$ and $K_{X_t}^2 = 2$ by using a standard argument.

We first prove that X_t satisfies $p_g = 0$ and $K^2 = 2$: Since \tilde{Z} is a nonsingular rational surface and X has only rational singularities, X is a projective surface with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then the upper semi-continuity implies $H^2(X_t, \mathcal{O}_{X_t}) = 0$, so that the Serre duality implies $p_g(X_t) = 0$ (Equivalently, it also follows from the fact that $p_g(X_t) = \frac{b_2^+(X_t) - 1}{2} = \frac{b_2^+(\tilde{Z}) - 1}{2} = 0$). And $K_X^2 = 2$ can be computed by using the explicit description of f^*K_X (refer to Equation (2) below). Then we have $K_{X_t}^2 = 2$ by the property of the \mathbb{Q} -Gorenstein smoothing.

Next, let us show the minimality of X_t : As we noticed in Section 3, the surface \tilde{Z} contains the following two chains of \mathbb{P}^1 including the proper transforms of three sections. We denote them by the following dual graphs

$$\begin{array}{cccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ G_1 & & G_2 & & G_3 & & G_4 & & G_5 & & G_6 & & G_7 & & G_8 \end{array}, \quad \begin{array}{cc} \circ & - & \circ \\ J_1 & & J_2 \end{array}$$

and we also denote four special fibers by the following dual graphs

$$\begin{array}{ccccccc}
& \tilde{A}_{\circ, -2} & & J_{2, -2} & & J_{\circ, -2} & \\
& \parallel & & & & \mid & \\
E'_{3, -1} & - \tilde{B}_{\circ, -4} & - E''_{3, -1}, & G_{3, -2} & - G_{4, -2} & - G_{5, -2} & - G_{6, -2} & - G_{7, -2}, \\
& & E'''_{1, -1} & & & & & \\
& & \mid & & & & & \\
& E'_{1, -1} & - I_{1, -7} & - I_{2, -2} & - I_{3, -2} & - I_{4, -2} & - E''_{1, -1} & \\
& & \parallel & & & & & \\
& & E_{1, -1}, & & & & & \\
& & E'''_{2, -1} & & & & & \\
& & \mid & & & & & \\
E'_{2, -1} & - H_{1, -2} & - H_{2, -7} & - H_{3, -2} & - H_{4, -2} & - H_{5, -3} & - E'_{2, -1} & - G_{1, -2} \\
& & \parallel & & & & & \\
& & E_{2, -1}. & & & & &
\end{array}$$

Note that the second one indicates the \tilde{E}_6 -singular fiber and J denotes the rational -2 -curve which is not contracted from \tilde{Z} to X . The numbers indicate the self intersection numbers of curves. Let $f : \tilde{Z} \rightarrow X$ and let $h : \tilde{Z} \rightarrow Y = E(1)$. Then we have

$$\begin{aligned}
K_{\tilde{Z}} \equiv & f^*K_X - \left(\frac{7}{15}G_1 + \frac{14}{15}G_2 + \frac{13}{15}G_3 + \frac{12}{15}G_4 + \frac{11}{15}G_5 + \frac{10}{15}G_6 + \frac{9}{15}G_7 + \frac{8}{15}G_8 + \frac{4}{9}H_1 \right. \\
& \left. + \frac{8}{9}H_2 + \frac{7}{9}H_3 + \frac{6}{9}H_4 + \frac{5}{9}H_5 + \frac{4}{5}I_1 + \frac{3}{5}I_2 + \frac{2}{5}I_3 + \frac{1}{5}I_4 + \frac{1}{2}B + \frac{2}{3}J_1 + \frac{1}{3}J_2 \right),
\end{aligned}$$

$$\begin{aligned}
K_{\tilde{Z}} \equiv & h^*K_Y + E_1 + E'_1 + 4E''_1 + E'''_1 + I_2 + 2I_3 + 3I_4 + E_2 + 2E'_2 + 8E''_2 + E'''_2 \\
& + H_1 + H_3 + 2H_4 + 3H_5 + 4G_1 + E'_3 + E'''_3
\end{aligned}$$

On the other hand,

$$\begin{aligned}
h^*K_Y \equiv & -\frac{1}{2}(2E_1 + E'_1 + E''_1 + E'''_1 + I_1 + I_2 + I_3 + I_4) \\
& -\frac{1}{2}(2E_2 + E'_2 + 2E''_2 + E'''_2 + H_1 + H_2 + H_3 + H_4 + H_5 + G_1)
\end{aligned}$$

Hence, combining these relations, we get

$$\begin{aligned}
f^*K_X \equiv & \frac{119}{30}G_1 + \frac{14}{15}G_2 + \frac{13}{15}G_3 + \frac{12}{15}G_4 + \frac{11}{15}G_5 + \frac{10}{15}G_6 + \frac{9}{15}G_7 + \frac{8}{15}G_8 + \frac{17}{18}H_1 \\
(2) \quad & + \frac{7}{18}H_2 + \frac{23}{18}H_3 + \frac{39}{18}H_4 + \frac{55}{18}H_5 + \frac{3}{10}I_1 + \frac{11}{10}I_2 + \frac{19}{10}I_3 + \frac{27}{10}I_4 + \frac{1}{2}B \\
& + \frac{2}{3}J_1 + \frac{1}{3}J_2 + \frac{1}{2}E'_1 + \frac{7}{2}E''_1 + \frac{1}{2}E'''_1 + \frac{3}{2}E'_2 + 7E''_2 + \frac{1}{2}E'''_2 + E'_3 + E'''_3.
\end{aligned}$$

Since all coefficients are positive in the expression of f^*K_X , the \mathbb{Q} -divisor f^*K_X is nef if $f^*K_X \cdot E'_i \geq 0$, $f^*K_X \cdot E''_i \geq 0$ for $i = 1, 2, 3$, and $f^*K_X \cdot E'''_i \geq 0$ for $i = 1, 2$. We have $f^*K_X \cdot E'_1 = \frac{7}{15}$, $f^*K_X \cdot E''_1 = \frac{2}{15}$, $f^*K_X \cdot E'''_1 = \frac{5}{15}$, $f^*K_X \cdot E'_2 = \frac{1}{9}$, $f^*K_X \cdot E''_2 = \frac{1}{45}$, $f^*K_X \cdot E'''_2 = \frac{19}{45}$, $f^*K_X \cdot E'_3 = \frac{1}{6}$, and $f^*K_X \cdot E'''_3 = \frac{13}{30}$. Note that other divisors are contracted under the map f . The nefness of f^*K_X implies the nefness of K_X . Since all coefficients are positive in the expression of f^*K_X , we get the vanishing $h^0(-K_X) = 0$. Hence, by the upper semi-continuity property, i.e. the vanishing $h^0(-K_X) = 0$ implies that $h^0(-K_{X_t}) = 0$, we conclude that X_t is not a rational surface: If X_t is a rational surface with $h^0(-K_{X_t}) = 0$, then $\chi(2K_{X_t}) \leq 0$. But $\chi(2K_{X_t}) = \chi(\mathcal{O}_{X_t}) + K_{X_t}^2 = 3$, which

is a contradiction. Since $K_{X_t}^2 = 2$, X_t is a surface of general type by the classification theory of surfaces. Let $\pi : \mathcal{X} \rightarrow \Delta$ be a \mathbb{Q} -Gorenstein smoothing of X . Since the \mathbb{Q} -Cartier divisor $K_{\mathcal{X}/\Delta}$ is π -big over Δ and π -nef at the point 0, the nefness of K_{X_t} is also obtained by shrinking Δ if it is necessary [28]. Therefore we have

Proposition 5.1. *X_t is a minimal surface of general type with $p_g = 0$ and $K_{X_t}^2 = 2$.*

Finally, applying the standard arguments about Milnor fibers ([23], §5), we prove Theorem 1.1: Note that X has five singularities, say p_1, \dots, p_5 . For each $1 \leq i \leq 5$, let (V_i, p_i) be a small disjoint neighborhood for the singularity p_i in X . By Proposition 2.5, there exists a closed embedding of (V_i, p_i) in $(\mathbb{C}^n, 0)$ satisfying that a local \mathbb{Q} -Gorenstein smoothing V of V_i is a closed embedding in $(\mathbb{C}^n \times \Delta, 0)$ and the map $\pi : V \rightarrow \Delta$ is induced by the projection on the second factor $\mathbb{C}^n \times \Delta \rightarrow \Delta$. Let $B_r = \{z \in \mathbb{C}^n \mid |z| < r\}$ and let $S_r = \partial B_r$. The sphere S_r is called a Milnor sphere for V_i if for every $0 < r' \leq r$ the sphere $S_{r'}$ intersects V_i transversally. Let S_r be a Milnor sphere for V_i then we can assume that $S_r \times \Delta$ intersects V_t transversally for all $t \in \Delta$ by shrinking Δ if it is necessary. In this set-up $F_i := V_t \cap (B_r \times \Delta)$ is called the Minor fiber of a \mathbb{Q} -Gorenstein smoothing. Then there are small disjoint neighborhoods (V_i, p_i) ($1 \leq i \leq 5$) for the five singularities p_i in X such that $\hat{X} := (X - \cup_{i=1}^5 V_i) \cup (\cup_{i=1}^5 F_i)$ is diffeomorphic to $X_t = (X - \cup_{i=1}^5 F_i) \cup (\cup_{i=1}^5 F_i)$, where the pasting is made by choosing an orientation preserving diffeomorphism $\partial V_i \rightarrow \partial F_i$ for each $1 \leq i \leq 5$. Note that, since these F_i are rational balls ([26], §1), the manifold $\hat{X} = (X - \cup_{i=1}^5 V_i) \cup (\cup_{i=1}^5 F_i)$ is diffeomorphic to the rational blow-down 4-manifold $\tilde{Z}_{15,9,5,3,2}$ constructed in Theorem 3.1. Hence the simple connectivity of X_t follows from the fact that $\tilde{Z}_{15,9,5,3,2}$ is simply connected.

6. MORE EXAMPLES

In this section we construct more examples of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $K^2 = 2$ by using different configurations coming from different elliptic pencils in \mathbb{P}^2 . Since all the proofs are basically the same as the case of the main example constructed in Section 3, we only explain how to construct more examples.

Construction. We first consider an elliptic fibration on $E(1)$ which has one \tilde{E}_7 -singular fiber (*III** in Kodaira's table of singular elliptic fibers in [4], p.201), three nodal fibers and three disjoint sections. Such an elliptic fibration can be constructed explicitly as follows ([38] for details): Let C_1 be a union of two lines L_1 and L_2 with the latter of multiplicity two, and let C_2 be an immersed 2-sphere with one positive transverse double point. Assume that L_2 intersects C_2 at a single point, say p , with triple tangency between two curves, and L_1 (also passing through p) intersects C_2 at two other smooth points, say q and r . Consider a cubic pencil in \mathbb{P}^2 induced by C_1 and C_2 , i.e. $\lambda C_1 + \mu C_2$, for $[\lambda : \mu] \in \mathbb{P}^1$. Now, blowing up seven times successively at the point p and blow up once at the points q and r respectively, we get a desired elliptic fibration $E(1) = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2$ over \mathbb{P}^1 . We denote this elliptic fibration by $g : Y' = E(1) \rightarrow \mathbb{P}^1$. Note that there is an \tilde{E}_7 -singular fiber coming from the total transforms of L_1 and L_2 , and three nodal fibers on Y' . Furthermore, the elliptic fibration Y' admits three disjoint sections, say S_1, S_2, S_3 , where S_1 is the 7th-exceptional curve and S_2, S_3 are the 8th- and the 9th-exceptional curves in the forming of Y' (Figure 3). Notice that the section S_1 meets an ending

2-sphere, say u_1 , of the \tilde{E}_7 -singular fiber, and two sections S_2 and S_3 meet the other ending 2-sphere, say u_7 , of the \tilde{E}_7 -singular fiber.

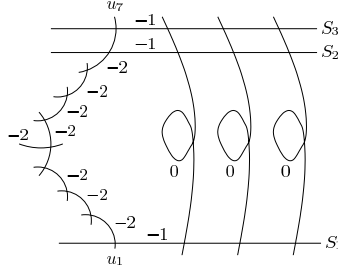
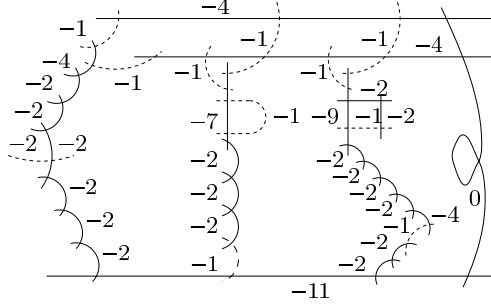


FIGURE 3. A rational surface Y'

Let $Z' := Y' \# 2\overline{\mathbb{P}}^2$ be the surface obtained by blowing up at two singular points of two nodal fibers on Y' . Then there are two fibers such that each consists of two \mathbb{P}^1 s, say E_i and F_i , satisfying $E_i^2 = -1$, $F_i^2 = -4$ and $E_i \cdot F_i = 2$ for $i = 1, 2$. Note that each E_i is an exceptional curve and F_i is the proper transform of a nodal fiber. First, we blow up six times at the intersection points between two sections S_2, S_3 and F_1, F_2, u_7 . It makes the self-intersection number of the proper transforms of S_2, S_3 and u_7 to be -4 , and the self-intersection number of the proper transforms of F_1 and F_2 are -6 . We also blow up four times successively at the intersection point between F_1 and the section S_1 which makes a chain of \mathbb{P}^1 , $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, and blow up six times successively at the intersection point between F_2 and S_1 which makes a chain of \mathbb{P}^1 , $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$. We further blow up twice successively at one intersection point between -7 -curve in the total transform of F_2 and the exceptional curve intersecting -7 -curve twice, and finally blow up twice successively at the intersection point between the ending -2 -curve and the exceptional curve in the total transform of F_2 , so that it makes a chain of \mathbb{P}^1 , $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-9}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$. Hence we have a rational surface $\tilde{Z}' := Y' \# 22\overline{\mathbb{P}}^2$ which contains five disjoint linear chains of \mathbb{P}^1 : $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-11}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$ (which contains the proper transforms of S_1 , the six rational -2 -curves in \tilde{E}_7 -singular fiber and the proper transform of u_7), $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-9}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$ (which is a part of the total transform of F_2), $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which is a part of the total transform of F_1), and two -4 -curves which are the proper transforms of S_2 and S_3 (Figure 4).

Next, we contract these five disjoint chains of \mathbb{P}^1 from \tilde{Z}' . This creates a projective surface X with five singularities of class T . The existence of a \mathbb{Q} -Gorenstein smoothing for X is obtained by the same proof as in Section 4. Let us denote a general fiber of the \mathbb{Q} -Gorenstein smoothing by X_t . Then, by the same argument in Section 5, we see that X_t is a minimal surface of general type with $p_g = 0$ and $K^2 = 2$. Finally, since the rational blow-down 4-manifold $\tilde{Z}'_{25,19,5,2,2}$ of the surface \tilde{Z}' is simply connected, which can be proved in a similar way as in the proof of Theorem 3.1, we conclude that X_t is a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 2$.

Remark. J. Keum suggested that one can also construct another example of simply connected surface of general type with $p_g = 0$ and $K^2 = 2$ by using an elliptic fibration on $E(1)$ which has one \tilde{D}_6 -singular fiber, one I_2 -singular fiber and two nodal fibers, and

FIGURE 4. A rational surface \tilde{Z}'

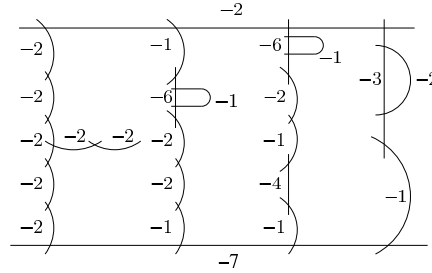
which also admits four disjoint sections [16]. Of course, the remaining proofs are the same as in Section 4 and 5. We do not know whether all these constructions above provide the same deformation equivalent type of surfaces with $p_g = 0$ and $K^2 = 2$. It is a very intriguing problem to determine whether the examples constructed above are diffeomorphic (or deformation equivalent) to the surface constructed in Section 3.

7. APPENDIX: SIMPLY CONNECTED SURFACES WITH $p_g = 0$ AND $K^2 = 1$

As mentioned in the Introduction, the only previously known simply connected complex surface of general type with $p_g = 0$ was Barlow surface, which has $K^2 = 1$. Dolgachev-Werner [9] constructed a minimal surface of general type with $p_g = 0$ and $K^2 = 1$ by using a special quintic surface with one elliptic singular point, which was founded by Craighero-Gattazzo [7]. Their surface has a trivial first homology group, but the simple connectivity is still unknown. More examples and systematic study of numerical Godeaux surfaces, i.e. the minimal surfaces of general type with $p_g = 0$ and $K^2 = 1$, are given in [6, 37]. In this appendix we construct simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $K^2 = 1$ using a rational blow-down surgery and a \mathbb{Q} -Gorenstein smoothing theory. Since all proofs are basically the same as the case of $K^2 = 2$, we only explain how to construct such examples.

Construction A1. We begin with the rational elliptic surface $Y = \mathbb{P}^2 \# 9\overline{\mathbb{P}}^2$ with special fibers used in Section 3. We use the same notations for $Z, F_1, F_2, E_1, E_2, S_1, S_2, \tilde{A}, \tilde{B}$ as in the main construction in Section 3. First, we blow up at the intersection points between F_1 and the section S_1 , so that the self-intersection number of the proper transform of S_1 is -2 . We also blow up at the intersection points between F_1, F_2, \tilde{B} and the section S_2 . It makes the self-intersection number of the proper transform of S_2 to be -4 , and the self-intersection number of the proper transform of \tilde{B} to be -3 . Next, we blow up twice successively at the intersection point between the proper transform of S_2 and the exceptional curve in the total transform of F_1 , so that it makes a chain of $\mathbb{P}^1, \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_1 . And we blow up at the intersection point between the proper transform of F_2 and the exceptional curve which intersects the proper transform of the section S_2 . That makes a chain of $\mathbb{P}^1, \overset{-6}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_2 . Then we blow up again at the intersection point between the exceptional -1 -curve and the rational -2 -curve, and blow up again at the intersection point between the rational -3 -curve, which is the proper transform of the rational -2 -curve, and the

proper transform of the section S_2 . It makes a chain of \mathbb{P}^1 , $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-4}{\circ} - \overset{-1}{\circ}$. Note that this process makes the self-intersection number of the proper transform of S_2 to be -7 . Finally, we have a rational surface $\tilde{Z} := Y \sharp 11\overline{\mathbb{P}}^2$ which contains four disjoint linear chains of \mathbb{P}^1 : $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which contains the proper transform of S_2 and a part of \tilde{E}_6 -fiber), $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, $\overset{-2}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ (which contains the proper transforms of S_1 and of \tilde{B}), and $\overset{-4}{\circ}$ (Figure 5).

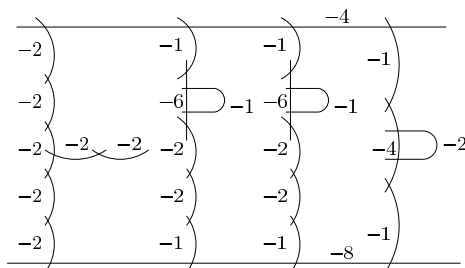
FIGURE 5. A rational surface \tilde{Z}

Next, by contracting these four disjoint chains of \mathbb{P}^1 from \tilde{Z} , we get a projective surface X with four singularities of class T . Then, by the same argument in Section 4 and 5 above, we finally see that the surface X has a \mathbb{Q} -Gorenstein smoothing and a general fiber X_t of the \mathbb{Q} -Gorenstein smoothing for X is a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 1$.

Construction A2. Similar to Construction A1 above, we use the same elliptic fibration on Y and we use the same notations for $Z, F_1, F_2, E_1, E_2, S_1, S_2, \tilde{A}, \tilde{B}$ as in the main construction in Section 3. First, we blow up at the intersection points between F_1, F_2, \tilde{B} and the sections S_1, S_2 respectively, so that the self-intersection numbers of the proper transforms of S_1 and S_2 are -4 . And we blow up twice successively at each intersection point between the proper transform of S_2 and the exceptional curves in the total transforms of F_1, F_2 , so that it makes two disjoint linear chains of \mathbb{P}^1 , $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ and $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transforms of F_1 and F_2 respectively. Note that this process makes the self-intersection number of the proper transform of S_2 to be -8 . Then, using four consecutive 2-spheres in \tilde{E}_6 -fiber which is not connected to the section S_1 together with the proper transform of S_2 , we also get a linear chain of \mathbb{P}^1 , $\overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$. Hence we finally have a rational surface $\tilde{Z}' := Y \sharp 12\overline{\mathbb{P}}^2$ which contains five disjoint linear chains of \mathbb{P}^1 : $\overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which contains the proper transform of S_2 and a part of \tilde{E}_6 -fiber), $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, $\overset{-4}{\circ}$ (which is the proper transform of S_1) and $\overset{-4}{\circ}$ (which is the proper transform of \tilde{B}) (Figure 6).

Then, as the same way in the Construction A1 above, we get a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 1$.

Open Problem. Determine whether the examples constructed in the Appendix above are diffeomorphic (or deformation equivalent) to the Barlow surface.

FIGURE 6. A rational surface \tilde{Z}'

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DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SINSU-DONG, MAPO-GU, SEOUL 121-742, KOREA

E-mail address: ynlee@sogang.ac.kr

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SAN 56-1, SILLIM-DONG, GWANAK-GU, SEOUL 151-747, KOREA

E-mail address: jipark@math.snu.ac.kr